Worcester County Mathematics League

Varsity Meet 4 – March 1, 2023

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

Worcester County Mathematics League

Varsity Meet 4 - March 1, 2023

Answer Key

Round 1 - Elementary Number Theory

1. 144
2. 75
3. 248

Round 2 - Algebra I

1. $\{-9, 9\}$ either order, with or without $\{\ \}$
2. 50
3. $(2, 3)$ exact order

Round 3 - Geometry

1. 60
2. $(2, 51)$ exact order
3. $(12, 3)$ exact order

Round 4 - Logs, Exponents, and Radicals

1. $\left(2, \dfrac{1}{60}\right)$ or $(2, 0.01\overline{6})$ exact order
2. $\{0, 2\}$ either order, with or without $\{\ \}$
3. 81

Round 5 - Trigonometry

1. $\sin x$
2. $\dfrac{58}{9}$ or $6\dfrac{4}{9}$ or $6.\overline{4}$
3. $(4, 4, 3)$ exact order

Team Round

1. 14
2. 800
3. $(12, 6)$ exact order
4. $(4, 3)$ exact order
5. $(2, 7)$ exact order
6. $0.\overline{2}$ or $0.\overline{2}_5$
7. 45
8. $(-23, 2, 6)$ exact order
9. $\cos x$

Worcester County Mathematics League
Varsity Meet 4 - March 1, 2023
Round 1 - Elementary Number Theory

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Two Worcester traffic lights turn red at the same time. One light turns red every 36 seconds and the other turns red every 48 seconds. How many seconds will ellapse before the two lights turn red at the same time again?

2. How many fewer digits will the base 16 representation of $n = 2^{100}$ have compared with the base 2 representation of the same number?

3. Given $n = ABC_9 = CBA_7$, that is, the representation of positive integer n in base 9 consists of the digits of its representation in base 7, only in reverse order. What is the base 10 representation of n?

ANSWERS

(1 pt) 1. ______________________ seconds

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Tahanto, Athol, Westborough

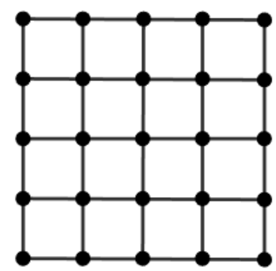
Worcester County Mathematics League
Varsity Meet 4 - March 1, 2023
Round 2 - Algebra I

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Solve for x:

$$-2x^2 + 8 = -154$$

2. Determine the total number of distinct squares that can be formed using vertices shown on the four by four checkerboard shown at right.



3. Solve the system of equations below for x and y. Express your answer as the ordered pair (x, y).

$$\frac{1}{5x} + \frac{1}{2y} = \frac{8}{5xy}$$
$$\frac{1}{3y} - \frac{1}{5x} = \frac{1}{15xy}$$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

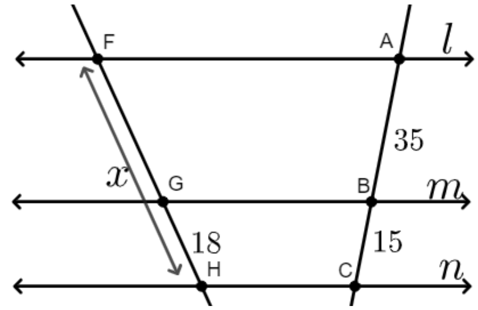
(3 pts) 3. $(x, y) =$ (________________)

St. Peter Marian, St. Peter Marian, Auburn

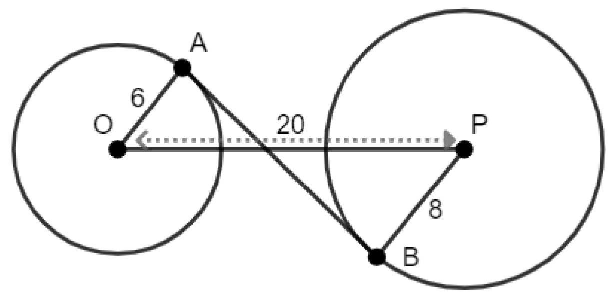
Worcester County Mathematics League
Varsity Meet 4 - March 1, 2023
Round 3 - Geometry

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

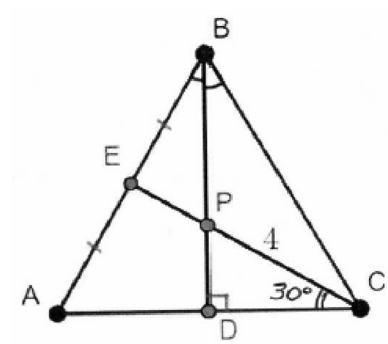
1. In the figure shown at right, lines $l \parallel m \parallel n$, $AB = 35$, $BC = 15$, and $GH = 18$. Find x, that is, FH.



2. $\overline{AB}$ is an internal tangent of circles O and P, as shown at right. Given that $OA = 6$, $PB = 8$, and $OP = 20$, AB can be expressed in simplest radical form as $a\sqrt{b}$. Find the ordered pair (a, b).



3. In ΔABC shown at right, $\overline{AE} \cong \overline{EB}$, $\angle ABD \cong \angle DBC$, $\overline{BD} \perp \overline{AC}$, $m\angle DCP = 30°$, and $CP = 4$. The area of ΔABC can be expressed in simplest radical form as $a\sqrt{b}$. Find the ordered pair (a, b).



ANSWERS

(1 pt) 1. $FH =$ ________________

(2 pts) 2. $(a, b) =$ (________________)

(3 pts) 3. $(a, b) =$ (________________)

Clinton, Hudson, Holliston

Worcester County Mathematics League
Varsity Meet 4 - March 1, 2023
Round 4 - Logs, Exponents, and Radicals

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. The expression $\sqrt[5]{\sqrt[4]{\sqrt[3]{2}}}$ can be written in the form a^b, where a is an integer with no perfect square, cube, or higher order factors. Find the ordered pair (a, b).

2. Find all values of x that solve the following equation:

$$3^{2x} - 10 \cdot 3^x + 9 = 0$$

3. Find all values of x that satisfy the following equation:

$$\log_3(\log_9 x) = \log_9(\log_3 x)$$

ANSWERS

(1 pt) 1.$(a, b) =$ (________________)

(2 pts) 2. ____________________

(3 pts) 3. ____________________

Tahanto, Shrewsbury, Hopkinton

Worcester County Mathematics League
Varsity Meet 4 - March 1, 2023
Round 5 - Trigonometry

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Express the following expression as a single trigonometric function of x:

$$\frac{\sec x}{\tan x + \cot x}$$

2. Given that $\tan A = \frac{3}{7}$, find:

$$\frac{\csc A \cdot \cot A}{\cos A}$$

3. In triangle ΔABC (not shown), $m\angle A = 60°$, $m\angle B = 45°$, $m\angle C = 75°$, and $AC = 8$. AB can be expressed in simplest radical form as $a + b\sqrt{c}$. Find the ordered triple (a, b, c).

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________

(3 pts) 3. $(a, b, c) =$ (____________)

Tantasqua, Quaboag, Bromfield

Worcester County Mathematics League
Varsity Meet 4 - March 1, 2023
Team Round

All answers must be in simplest exact form in the answer section.
NO CALCULATORS ALLOWED

1. Find the largest n such that 3^n is a factor of 30!.

2. Amy has \$1300. She invests part of her money at 8% annual interest rate and the rest at 5% annual interest rate. If she earned \$89 in interest after the first year, how much did Amy invest at 8%?

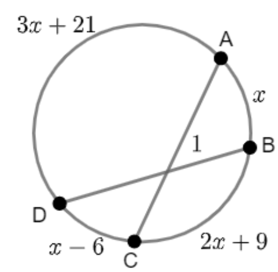
3. Semicircles are drawn on the three sides of a right triangle (not shown), where the diameter of each semicircle coincides with a side of the triangle. The semicircles have areas $6\pi, 9\pi,$ and 15π. The area of the triangle can be expressed in simplest radical form as $a\sqrt{b}$. Find the ordered pair (a, b).

4. A bacteria colony has an initial population of $3 \cdot 10^8$ and grows exponentially with time. If population is $9 \cdot 10^8$ after two hours of growth, then the number of hours required for the population to double in size can be expressed in the form $\frac{\ln a}{\ln b}$ where a and b are integers and b has no square factors other than 1. Find the ordered pair (a, b).

5. Parallelogram GRAM (not shown) has side lengths $GR = 4$ and $GM = 6$. Also, $m\angle G = 60°$. If the length of diagonal RM, expressed in simplest radical form, equals $a\sqrt{b}$, find the ordered pair (a, b).

6. What is the base 5 representation of one half (0.5 in base 10, that is, decimal notation)? Note that $(0.1)_5 = 5^{-1} = \frac{1}{5}$ or 0.2 in base 10, and $(0.01)_5 = 5^{-2} = \left(\frac{1}{5}\right)^2 = \frac{1}{25}$, or 0.04 in base 10.

7. Given the figure at right with m $\overset{\frown}{AB} = x°$, m $\overset{\frown}{BC} = 2x + 9°$, m $\overset{\frown}{CD} = x - 6°$, and m $\overset{\frown}{AD} = 3x + 21°$, find the measure of $\angle 1$ in degrees.



8. The following expression is equal to $\frac{a\sqrt{b}}{c}$ when written in simplest radical form.

$$5\sqrt{\frac{2}{9}} + 3\sqrt{\frac{9}{2}} - 5\sqrt{8}$$

Find the ordered triple (a, b, c) where $c > 0$.

9. Express the following expression as a single trigonometric function of x:

$$2\sin\frac{x}{2}\cos\frac{x}{2}\cot x$$

Mass Acad., Milbury, Algonquin, Leicester, Doherty, Doherty, St. John's, West Boylston, Northbridge

Worcester County Mathematics League
Varsity Meet 4 - March 1, 2023
Team Round Answer Sheet

ANSWERS

1. ______________________

2. \$ ______________________

3. $(a, b) =$ (____________________)

4. $(a, b) =$ (____________________)

5. $(a, b) =$ (____________________)

6. ______________________

7. ______________________ $^\circ$

8. $(a, b, c) =$ (____________________)

9. ______________________

Mass Acad., Milbury, Algonquin, Leicester, Doherty, Doherty, St. John's, West Boylston, Northbridge

Worcester County Mathematics League
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3. $(2, 3)$ exact order

Round 3 - Geometry

1. 60
2. $(2, 51)$ exact order
3. $(12, 3)$ exact order

Round 4 - Logs, Exponents, and Radicals

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9. $\cos x$

Round 1 - Elementary Number Theory

1. Two Worcester traffic lights turn red at the same time. One light turns red every 36 seconds and the other turns red every 48 seconds. How many seconds will ellapse before the two lights turn red at the same time again?

Solution: Let m be the number of times the first traffic turns red, and n be the number of times the second traffic turns red at t seconds, when the two lights next turn red at the same time. Then $t = 36m = 48n$. Divide each side of this equation by 12, and $3m = 4n$. The smallest positive integer values of m and n that solve this equation are $m = 4$ and $n = 3$. Thus, both lights turn red simultaneously after $t = 4 \cdot 36 = 3 \cdot 48 = \boxed{144}$ seconds. Note that 144 is the Least Common Multiple of 36 and 48.

2. How many fewer digits will the base 16 representation of $n = 2^{100}$ have compared with the base 2 representation of the same number?

Solution: Note that a number represented in base b as $d_k d_{k-1} ... d_1 d_0$ is equal to:

$$(d_k d_{k-1} ... d_1 d_0)_b = d_k \cdot b^k + d_{k-1} \cdot b^{k-1} + ... + d_1 \cdot b^1 + d_0 \cdot b_0.$$

The binary (base 2) representation of 2^{100} is therefore a 1 followed by 100 zeros, since only one power of 2 (2^{100}) is needed to represent the number. It has 101 binary digits.

Note that $16 = 2^4$, and that $2^{100} = 2^{4 \cdot 25} = (2^4)^{25} = 16^{25}$. The base 16 representation of 2^{100} therefore has 26 digits, a 1 followed by 25 zeros, since $2^{100} = 1 \cdot 16^{25} + 0 \cdot 16^{24} + 0 \cdot 16^{23} + ... + 0 \cdot 16 + 0 \cdot 1$. The difference in the number of digits is therefore $101 - 26 = \boxed{75}$.

3. Given $n = ABC_9 = CBA_7$, that is, the representation of positive integer n in base 9 consists of the digits of its representation in base 7, only in reverse order. What is the base 10 representation of n?

Solution: From the problem statement, A, B, and C must be *both* base 9 and base 7 digits. Therefore they must be equal to 0, 1, 2, 3, 4, 5, or 6; any other integer is not a base 7 digit. Expand the original equation in the powers of the two bases and collect terms:

$$\begin{aligned} ABC_9 &= CBA_7 \\ A \cdot 9^2 + B \cdot 9 + C &= C \cdot 7^2 + B \cdot 7 + A \\ 81A + 9B + C &= 49C + 7B + A \\ (9-7)B &= (49-1)C + (1-81)A \\ 2B &= 48C - 80 \end{aligned}$$

Divide both sides by 2 and $B = 24C - 40A = 8(3C - 5A)$. Therefore either 8 is a factor of B, or $B = 0$. Since B is no larger than 6, it does not have 8 as a factor, and $B = 0$. Then since B equals 0, $3C - 5A = 0$, and $3C = 5A$. Therefore $C = 5$ and $A = 3$, which is the only solution given that A and C must be integers between 0 and 6, inclusive. Finally, $ABC_9 = 305_9 = 3 \cdot 81 + 5 = CBA_7 = 503_7 = 5 \cdot 49 + 3 = \boxed{248}$.

Round 2 - Algebra I

1. Solve for x:

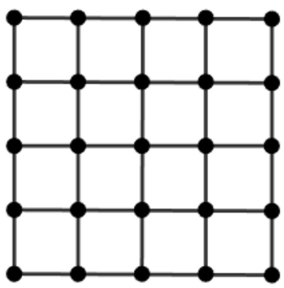
$$-2x^2 + 8 = -154$$

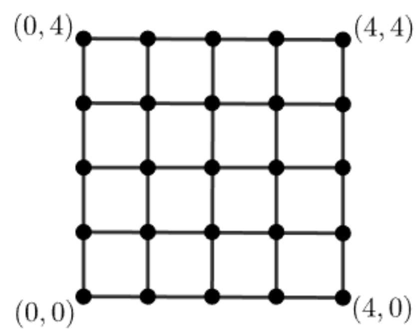
Solution: First add $2x^2 - 8$ to both sides of the equation, then divide all terms by 2:

$$\begin{aligned} -2x^2 + 8 &= -154 \\ 0 &= 2x^2 - 8 - 154 = 2x^2 - 162 \\ 0 &= x^2 - 81 = x^2 - 9^2 \\ 0 &= (x-9)(x+9) \end{aligned}$$

so $x - 9 = 0$ or $x + 9 = 0$, and $\boxed{\text{x = 9 or x = -9}}$.

2. Determine the total number of distinct squares that can be formed using vertices shown on the four by four checkerboard shown below, left.





Solution: The problem asks for the number of squares formed using vertices from the figure, but does not specify that they must be formed using line segments shown in the figure. Such squares have a number of different sizes and orientations and will be counted case by case. To facilitate the counting, number the vertices with horizontal (x) and vertical (y) coordinates from 0 to 4. The (x, y) coordinates for the four corner vertices are shown on the figure above, right, with the lower left vertex having coordinates $(0, 0)$. If a vertex has coordinates (m, n), then the vertex to its right will have coordinates $(m + 1, n)$ and the vertex above it will have coordinates $(m, n + 1)$.

There are four sizes of squares formed of both vertices and line segments from the figure. Consider the square with a side length of 1 ("unit square") whose lower left vertex has coordinates $(0, 0)$. This square can be translated $0, 1, 2$ or 3 units horizontally and $0, 1, 2$ or 3 units vertically to concide with any unit square in the checkerboard figure. There are four options for horizontal translation and four options for vertical translation, so there are $4 \cdot 4 = 16$ unit squares. Likewise, the square of side length 2 can be translated $0, 1,$ or 2 units horizonally and vertically for $3 \cdot 3 = 9$ squares of side length 2. There are $2 \cdot 2 = 4$ squares of side length 3, and exactly 1 square (the entire checkerboard) of side length 4. In total, there are $16 + 9 + 4 + 1 = 30$ squares formed from the vertices and line segments of the figure.

Next, consider squares that are "tilted". The square with vertices $(1, 0), (2, 1), (1, 2)$ and $(0, 1)$ is tilted at an angle of 45° and has side length $\sqrt{2}$. That square can be translated $0, 1,$ or 2 units horizontally and vertically, so there are $3 \cdot 3 = 9$ squares of this size and orientation. There is exactly one other square tilted at 45°; it has side length $2\sqrt{2}$ and vertices $(2, 0), (4, 2), (2, 4),$ and $(0, 2)$. There is a tilted square with vertices $(1, 0), (3, 1), (2, 3)$ and $(0, 2)$ that has side length $\sqrt{5}$; it can be translated 0 or 1 units horizontally and vertically, so there are $2 \cdot 2 = 4$ squares of this size and orientation. There is likewise a tilted square with vertices $(2, 0), (3, 2), (1, 3)$ and $(0, 1)$ that has side length $\sqrt{5}$; it can be translated similarly and there are 4 squares of this size and orientation. Finally, there are two tilted squares of side length $\sqrt{10}$ units: one has vertices $(1, 0), (4, 1), (3, 4)$ and $(0, 3)$; the other has vertices $(3, 0), (4, 3), (1, 4)$ and $(0, 1)$. In all there are $9 + 1 + 4 + 4 + 2 = 20$ tilted squares. Together with the 30 untilted squares there are $30 + 20 = \boxed{50}$ total squares.

3. Solve the system of equations below for x and y. Express your answer as the ordered pair (x, y).

$$\begin{aligned}\frac{1}{5x} + \frac{1}{2y} &= \frac{8}{5xy}\\ \frac{1}{3y} - \frac{1}{5x} &= \frac{1}{15xy}\end{aligned}$$

Solution: Different methods will solve this system of equations. Perhaps the easiest method is to add the rational expressions on the left hand side of each equation, find a common denominator with the right hand side, and equate the numerators on either side of the equation:

$$\frac{1}{5x} + \frac{1}{2y} = \frac{2y + 5x}{(5x)(2y)} = \frac{5x + 2y}{10xy} = \frac{8}{5xy} = \frac{16}{10xy}, \text{ so } 5x + 2y = 16$$
$$\frac{1}{3y} - \frac{1}{5x} = \frac{5x - 3y}{(5x)(3y)} = \frac{5x - 3y}{15xy} = \frac{1}{15xy}, \text{ so } 5x - 3y = 1$$

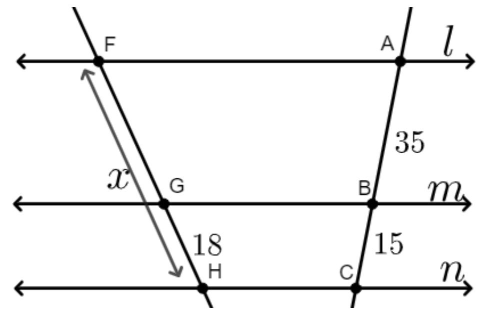
The two equations on the right form a linear system of equations equivalent to the original system. Solve the linear system by subtracting the second equation from the first:

$$\begin{aligned}5x + 2y &= 16\\ -(5x - 3y &= 1)\\ 5y &= 15\end{aligned}$$

So $y = \frac{15}{5} = 3$. Substitute for y in the first equation $5x + 2 \cdot 3 = 16$ and solve for x: $5x = 16 - 6 = 10$, so $x = \frac{10}{5} = 2$. The ordered pair $(x, y) = \boxed{(2, 3)}$.

Round 3 - Geometry

1. In the figure shown at right, lines $l \parallel m \parallel n$, $AB = 35$, $BC = 15$, and $GH = 18$. Find x, that is, FH.

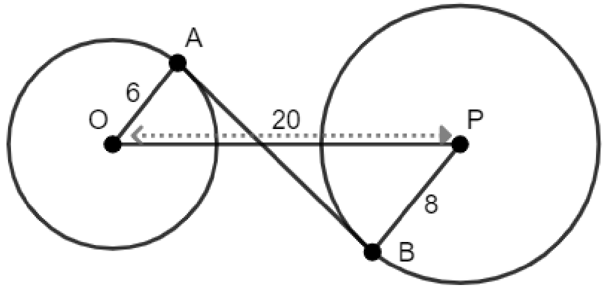


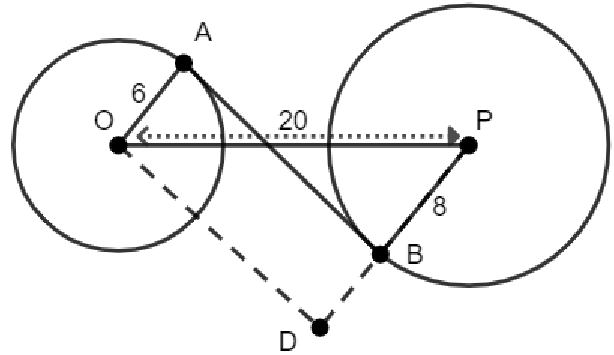
Solution: The figure shows three parallel lines and two transversals. It is a fact that three parallel lines that divide a transversal in a ratio $a : b$ will divide all other transversals in the same ratio. That fact is proved for this diagram by constructing the line segment $\overline{AH}$ and applying the Triangle Proportionality Theorem to ΔAHC and ΔAHF.
Solve for x by first writing the proportion, with $FG = x - 18$:

$$\frac{FG}{GH} = \frac{AB}{BC} = \frac{x-18}{18} = \frac{35}{15} = \frac{7}{3}$$

Multiply the third and last ratios by 18: $x - 18 = \frac{18}{3} \cdot 7 = 6 \cdot 7 = 42$, and $x = 42 + 18 = \boxed{60}$.

2. $\overline{AB}$ is an internal tangent of circles O and P, as shown below, left. Given that $OA = 6$, $PB = 8$, and $OP = 20$, AB can be expressed in simplest radical form as $a\sqrt{b}$. Find the ordered pair (a, b).



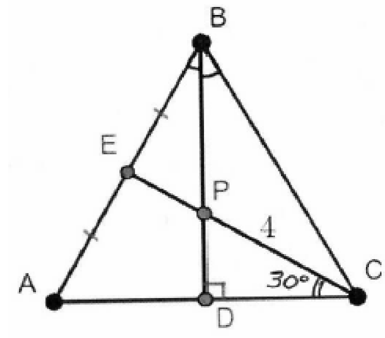


Solution: The most direct method to solve this problem is to construct a right triangle whose hypotenuse is $\overline{OP}$. First note that $\overline{OA} \perp \overline{AB}$ and $\overline{PB} \perp \overline{AB}$ because a radius drawn to the point of tangency is perpendicular to the tangent. Next, construct the line segment through O parallel to $\overline{AB}$ and extend radius $\overline{PB}$ until it intersects the constructed line segment at point D, as shown in the figure above on the right. Then $\overline{BD} || \overline{OA}$, and $ABDO$ is a rectangle. Therefore $\angle D$ is a right angle, and ΔODP is a right triangle. Now $BD = OA = 6$ (they are opposite sides of the rectangle) and $PD = PB + BD = 8 + 6 = 14$. Also, it was given that $OP = 20$. Apply the Pythagorean Theorem to find OD: $OD^2 + PD^2 = OP^2$, or:

$$OD^2 = OP^2 - PD^2 = 20^2 - 14^2 = 400 - 196 = 204$$

Thus, $AB = OD = \sqrt{204} = \sqrt{4 \cdot 51} = \sqrt{4}\sqrt{51} = 2\sqrt{51}$ and $(a, b) = \boxed{(2, 51)}$.

3. In ΔABC shown at right, $\overline{AE} \cong \overline{EB}$, $\angle ABD \cong \angle DBC$, $\overline{BD} \perp \overline{AC}$, m$\angle DCP = 30°$, and $CP = 4$. The area of ΔABC can be expressed in simplest radical form as $a\sqrt{b}$. Find the ordered pair (a, b).



Solution: Note that $\overline{BD}$ is both an altitude and an angle bisector of ΔABC. Therefore it divides ΔABC into congruent triangles: $\angle ABD \cong \angle CBD, \overline{BD} \cong \overline{BD}, \angle ADB \cong \angle CDB$, so $\Delta ABD \cong \Delta CBD$ by ASA. Then $\overline{AD} \cong \overline{DC}$ and $\overline{AB} \cong \overline{BC}$ because they are corresponding parts of congruent triangles. Thus $\overline{BD}$ is a median, and ΔABC is an isosceles triangle.
Next, note that ΔPDC is a 30-60-90 triangle (because m$\angle PCD = 30°$ and m$\angle PDC = 90°$) and its sides are in proportion $PD : DC : CP = 1 : \sqrt{3} : 2 = 2 : 2\sqrt{3} : 4$. Thus, $PD = 2$ and $DC = 2\sqrt{3}$. Now, recall that the medians of any triangle divide each other in the ratio $1 : 2$, so $DP : PB = 1 : 2 = 2 : 4$, and $PB = 4$.
Now the area of ΔABC is equal to $\frac{1}{2}bh$, where h is the height of the triangle and b is the base length. In this case, $h = BD = DP + PB = 2 + 4 = 6$ and $b = AC = AD + DC = 2\sqrt{3} + 2\sqrt{3} = 4\sqrt{3}$. The area of ΔABC is $\frac{1}{2}bh = \frac{1}{2} \cdot 6 \cdot 4\sqrt{3} = 3 \cdot 4\sqrt{3} = 12\sqrt{3}$. Thus, $(a, b) = \boxed{(12, 3)}$.

Round 4 - Logs, Exponents, and Radicals

1. The expression $\sqrt[5]{\sqrt[4]{\sqrt[3]{2}}}$ can be written in the form a^b, where a is an integer with no perfect square, cube, or higher order factors. Find the ordered pair (a, b).

Solution: Note that a radical function can be expressed as an expontial function. This fact is demonstrated by noting that $(\sqrt[n]{a})^n = a$ (because $\sqrt[n]{a}$ is the root of the equation $x^n = a$) and also noting that $a = a^1$. Then if we replace $\sqrt[n]{a}$ by a^b, we find that $\left(a^b\right)^n = a^{bn} = a^1$. Equating the exponents, $bn = 1$, or $b = \frac{1}{n}$. Thus,

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

.

Now, apply this identity to $\sqrt[5]{\sqrt[4]{\sqrt[3]{2}}}$ three times, working from the innermost radical towards the outer radical:

$$\begin{aligned} \sqrt[3]{2} &= 2^{\frac{1}{3}} \\ \sqrt[4]{\sqrt[3]{2}} &= \sqrt[4]{2^{\frac{1}{3}}} = \left(2^{\frac{1}{3}}\right)^{\frac{1}{4}} = 2^{\frac{1}{3}\cdot\frac{1}{4}} = 2^{\frac{1}{12}} \\ \sqrt[5]{\sqrt[4]{\sqrt[3]{2}}} &= \sqrt[5]{2^{\frac{1}{12}}} = \left(2^{\frac{1}{12}}\right)^{\frac{1}{5}} = 2^{\frac{1}{12}\cdot\frac{1}{5}} = 2^{\frac{1}{60}} \end{aligned}$$

and $(a, b) = \boxed{\left(2, \frac{1}{60}\right)}$.

2. Find all values of x that solve the following equation:

$$3^{2x} - 10 \cdot 3^x + 9 = 0$$

Solution: Let $y = 3^x$, substitute y for 3^x into the equation and factor the quadratic:

$$\begin{aligned} 3^{2x} - 10 \cdot 3^x + 9 &= (3^x)^2 - 10 \cdot 3^x + 9 \\ &= y^2 - 10y + 9 \\ &= (y - 9)(y - 1) \end{aligned}$$

so $y = 9$ or $y = 1$. Substitute 3^x for y into each equation and solve for x. If $3^x = 9$ then $x = 2$. If $3^x = 1$ then $x = 0$, so $x = \boxed{0 \text{ or } 2}$.

3. Find all values of x that satisfy the following equation:

$$\log_3(\log_9 x) = \log_9(\log_3 x)$$

Solution: First, note that $\log_9 x$ can be written in terms of $\log_3 x$ by applying the change of basis formula and simplifying:

$$\log_9 x = \frac{\log_3 x}{\log_3 9} = \frac{\log_3 x}{\log_3 3^2} = \frac{\log_3 x}{2}$$

Next, apply this identity to the two base 9 logarithms, multiply both sides of the equation by 2, and apply the exponent law of logarithms:

$$\begin{aligned}
\log_3\left(\frac{\log_3 x}{2}\right) &= \frac{\log_3(\log_3 x)}{2} \\
2\log_3\left(\frac{\log_3 x}{2}\right) &= \log_3(\log_3 x) \\
\log_3\left(\frac{\log_3 x}{2}\right)^2 &= \log_3(\log_3 x)
\end{aligned}$$

Next, note that both sides of the equation are base 3 logarithm functions. Equate the arguments of these functions, divide both sides by $\log_3 x$, and multiply both sides by 4:

$$\begin{aligned}
\left(\frac{\log_3 x}{2}\right)^2 &= \log_3 x \\
\frac{\log_3 x}{4} &= 1 \\
\log_3 x &= 4
\end{aligned}$$

Apply the inverse identity to this equation: $3^4 = x$, or $x = 3^4 = \boxed{81}$

Round 5 - Trigonometry

1. Express the following expression as a single trigonometric function of x:
$$\frac{\sec x}{\tan x + \cot x}$$

Solution: Begin by converting the three functions to functions of $\sin x$ and $\cos x$:
$$\frac{\sec x}{\tan x + \cot x} = \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}$$
Next, multiply both the numerator and denominator by $\sin x \cos x$:
$$\frac{\left(\frac{1}{\cos x}\right)\sin x \cos x}{\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)\sin x \cos x} = \frac{\sin x}{\sin^2 x + \cos^2 x}$$
Apply the Pythagorean identity ($\sin^2 x + \cos^2 x = 1$), and the expression reduces to $\boxed{\sin x}$.

2. Given that $\tan A = \frac{3}{7}$, find:
$$\frac{\csc A \cdot \cot A}{\cos A}$$

Solution: First, simplify the expression by applying the definitions of $\csc x$ $(= \frac{1}{\sin x})$ and $\cot x$ $(= \frac{\cos x}{\sin x})$, then recognizing that dividing by an expression ($\cos x$ in this case) is the same as multiplying by its recoprocal, and finally by cancelling terms:
$$\frac{\csc A \cdot \cot A}{\cos A} = \frac{\frac{1}{\sin A} \cdot \frac{\cos A}{\sin A}}{\cos A} = \frac{1}{\sin A} \cdot \frac{\cos A}{\sin A} \cdot \frac{1}{\cos A} = \frac{1}{\sin^2 A}$$
Now to find $\sin^2 A$, note that $\tan A = \frac{3}{7}$, so $\tan^2 A = \frac{\sin^2 A}{\cos^2 A} = \left(\frac{3}{7}\right)^2 = \frac{9}{49}$, and that $\cos^2 A = 1 - \sin^2 A$, so
$$\tan^2 A = \frac{\sin^2 A}{1 - \sin^2 A} = \frac{9}{49}.$$
Cross multiply the last equation to solve for $\sin^2 A$:
$$49 \sin^2 A = 9(1 - \sin^2 A) = 9 - 9\sin^2 A$$
$$49 \sin^2 A + 9 \sin^2 A = 58 \sin^2 A = 9$$
$$\sin^2 A = \frac{9}{58}$$
and $\frac{1}{\sin^2 A} = \boxed{\frac{58}{9}}$.

3. In triangle ΔABC (not shown), $m\angle A = 60°$, $m\angle B = 45°$, $m\angle C = 75°$, and $AC = 8$. AB can be expressed in simplest radical form as $a + b\sqrt{c}$. Find the ordered triple (a, b, c).

Solution: First, recall the Law of Sines for triangle ΔABC: $\frac{AB}{\sin C} = \frac{AC}{\sin B} = \frac{BC}{\sin A}$. We only need the equation of the first two ratios. Solving for AB and substituting values for $\angle C$ and $\angle B$:

$$AB = \sin C \frac{AC}{\sin B} = 8\left(\frac{\sin 75°}{\sin 45°}\right)$$

Recall that $\sin 45° = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$. To find $\sin 75°$, recall the angle sum formula $\sin x + y = \sin x \cos y + \sin y \cos x$ and set $x = 30°$ and $y = 45°$:

$$\begin{aligned}\sin 75° = \sin 30° + 45° &= \sin 30° \cos 45° + \sin 45° \cos 30° \\ &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{2}\cdot\sqrt{3}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

To find the answer, substitute the values for $\sin 45°$ (expressed as $\frac{1}{\sqrt{2}}$) and $\sin 75°$ into the expression for AB and simplify:

$$\begin{aligned}AB = 8\frac{\frac{\sqrt{2}+\sqrt{6}}{4}}{\frac{1}{\sqrt{2}}} &= \frac{8\sqrt{2}(\sqrt{2}+\sqrt{6})}{4} \\ &= 2(\sqrt{2}\cdot\sqrt{2} + \sqrt{2}\cdot\sqrt{6}) \\ &= 2(2 + \sqrt{2\cdot 2\cdot 3}) \\ &= 4 + 2\sqrt{4}\sqrt{3} \\ &= 4 + 2\cdot 2\sqrt{3} \\ &= 4 + 4\sqrt{3}\end{aligned}$$

and $(a, b, c) = \boxed{(4, 4, 3)}$.

Team Round

1. Find the largest n such that 3^n is a factor of 30!.

Solution: 30 factorial, or 30! is the product $30 \cdot 29 \cdot 28 \cdot ... \cdot 2 \cdot 1$. The question asks for the largest n such that 3^n is a factor of 30!. Then n is equal to the sum of the number of factors of 3 in each of the product terms. Ten of the terms $(3, 6, 9, ..., 30)$ have at least one factor of 3. Two of those terms (9, 18) have exactly two factors of three and one term (27) has three factors of 3. Thus, add 10 (the number of terms with a factor of 3) plus 2 for the two terms with one extra factor plus 2 for the term with two extra factors: $10 + 2 + 2 = \boxed{14}$.

2. Amy has \$1300. She invests part of her money at 8% annual interest rate and the rest at 5% annual interest rate. If she earned \$89 in interest after the first year, how much did Amy invest at 8%?

Solution: Let x equal the amount that Amy invested at 8%. One year's interest on that amount is $0.08x$. Then $\$1300 - x$ is the amount that Amy invested at 5% and one year's interest on this amount is $0.05(\$1300 - x)$. The total interest earned, \$89 is equal to the sum of these two amounts:

$$89 = .08x + 0.05(1300 - x) = (.08 - .05)x + 0.05 \cdot 1300 = (.03)x + 65$$

Solve for x:

$$x = \frac{89 - 65}{.03} = \frac{24}{.03} = \frac{2400}{3} = 800$$

and Amy invested $\$\boxed{800}$ at 8%.

3. Semicircles are drawn on the three sides of a right triangle (not shown), where the diameter of each semicircle coincides with a side of the triangle. The semicircles have areas $6\pi, 9\pi,$ and 15π. What is the area of the triangle?

Solution: The area of a semicircle with radius r is $\frac{\pi r^2}{2}$. If $d = 2r$ is the diameter, the area of the semicircle is $\dfrac{\pi\left(\frac{d}{2}\right)^2}{2} = \dfrac{\pi d^2}{8}$. Let the the three sides of the triangle be d_1, d_2, and d_3, the diameters of the three circles. Then $\frac{\pi d_1^2}{8} = 6\pi$, $\frac{\pi d_2^2}{8} = 9\pi$, and $\frac{\pi d_3^2}{8} = 15\pi$. Solve for the triangle sides, and $d_1 = \sqrt{48}$, $d_2 = \sqrt{72}$, and $d_3 = \sqrt{120}$. Note that $d_1^2 + d_2^2 = 48 + 72 = 120 = d_3^2$, confirming that the triangle is a right triangle. The area of a right triangle is half the product of its legs, so the area is $\frac{\sqrt{48}\sqrt{72}}{2} = \frac{\sqrt{16\cdot3}\sqrt{36\cdot2}}{2} = \frac{\sqrt{16}\cdot\sqrt{3}\sqrt{36}\cdot\sqrt{2}}{2} = \frac{4\sqrt{3}\cdot6\sqrt{2}}{2} = 4 \cdot 3\sqrt{3 \cdot 2} = \boxed{12\sqrt{6}}$.

4. A bacteria colony has an initial population of $3 \cdot 10^8$ and grows exponentially with time. If population is $9 \cdot 10^8$ after two hours of growth, then the number of hours required for the population to double in size can be expressed in the form $\frac{\ln a}{\ln b}$ where a and b are integers and b has no square factors other than 1. Find the ordered pair (a, b).

Solution: Let T_d be the number of hours that it takes the population to double, that is, the doubling time. Then the population at t hours can be represented by the exponential function

$$P(t) = P_0 \cdot 2^{\frac{t}{T_d}}$$

where P_0 is the population at time $t = 0$ and t is the elapsed time in hours. The given information is that the initial population is $P_0 = 3 \cdot 10^8$ and the population after two hours ($t = 2$) is $P(2) = 9 \cdot 10^8$. Substitute these values into the exponential function:

$$P(2) = 9 \cdot 10^8 = 3 \cdot 10^8 \cdot 2^{\frac{2}{T_d}}$$

Next, divide both sides of the equation by $3 \cdot 10^8$ and take the base 2 logarithm of each side:

$$3 = 2^{\frac{2}{T_d}}$$

$$\log_2 3 = \log_2 2^{\frac{2}{T_d}} = \frac{2}{T_d}$$

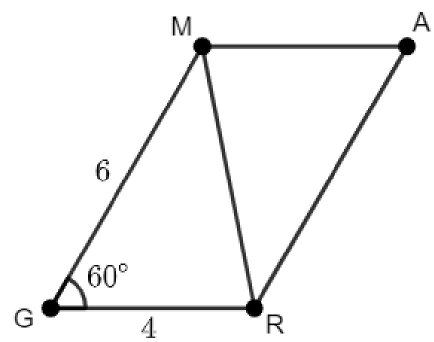
Multiply both sides of the equation by $\frac{T_d}{\log_2 3}$ and $T_d = \frac{2}{\log_2 3} = 2 \log_3 2$ (because $\frac{1}{log_a b} = \log_b a$). Finally, $T_d = 2 \log_3 2 = \log_3 2^2 = \log_3 4 = \frac{\ln 4}{\ln 3}$ after applying the exponent identity for logarithms and the change of basis formula, and $(a, b) = \boxed{(4, 3)}$

5. Parallelogram GRAM (not shown) has side lengths $GR = 4$ and $GM = 6$. Also, m$\angle G = 60°$. If the length of diagonal RM, expressed in simplest radical form, equals $a\sqrt{b}$, find the ordered pair (a, b).

Solution: First, draw the figure, shown at right. Next, note that two side lengths (GR, GM) and the measure of an included angle ($\angle G$) of ΔGRM are given information. Therefore the Law of Cosines can be applied to ΔGRM:

$$\begin{aligned} RM^2 &= GR^2 + GM^2 - 2(GR)(GM)\cos\angle G \\ &= 4^2 + 6^2 - 2(4)(6)\cos 60° \\ &= 16 + 36 - 48\frac{1}{2} \\ &= 52 - 24 = 28 \end{aligned}$$

Finally, $RM = \sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4}\sqrt{7} = 2\sqrt{7}$ and $(a, b) = \boxed{(2, 7)}$.



6. What is the base 5 representation of one half (0.5 in base 10, that is, decimal notation)? Note that $(0.1)_5 = 5^{-1} = \frac{1}{5}$ or 0.2 in base 10, and $(0.01)_5 = 5^{-2} = \left(\frac{1}{5}\right)^2 = \frac{1}{25}$, or 0.04 in base 10.

Solution: Note that the base 5 representation of one half (0.5_{10}) will be of the form $0.abc..._5$. That is, the "ones" digit will be zero because $0.5 < 1$. Then the base 5 series expansion of 0.5 will be:

$$0.5 = a \cdot 5^{-1} + b \cdot 5^{-2} + c \cdot 5^{-3} + ...$$

because the value of each digit in base 5 is $\frac{1}{5} = 5^{-1}$ times the digit to its left.

To find the first digit a, write the expansion as $0.5 = a \cdot 5^{-1} + r_1 = 0.2a + r_1$, where all numbers are decimals (base 10) and $r_1 < 0.2$. Then divide 0.5 by 5^{-1}, set a to the integer portion and set $5 \cdot r_1$ (the remainder) equal to the fractional portion. Note that dividing by 5^{-1} is the same as multiplying by 5, so:

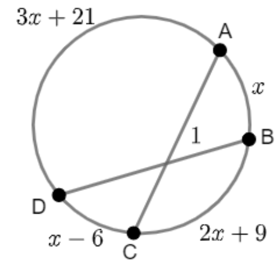
$$\begin{aligned} 5(0.5) &= 5(a \cdot 5^{-1} + r_1) \\ 2.5 &= a + 5 \cdot r_1. \end{aligned}$$

Therefore $a = 2$ and the remainder $5 \cdot r_1 = 0.5$. To find the next digit b, note that the first remainder $0.5 = 5r_1 = 5(b \cdot 5^{-2} + r_2)$. As in the previous step, divide this equation by 5^{-1}, that is multiply by 5, so that:

$$\begin{aligned} 5(0.5) &= 5 \cdot 5(b \cdot 5^{-2} + r_2) \\ 2.5 &= b + 5^2 \cdot r_2. \end{aligned}$$

Again, $b = 2$, the integer portion, and the remainder $5^2 \cdot r_2 = 0.5$. This process can be extended indefinitely. At the n^{th} step, after multiplying the remainder from the previous step (0.5) by 5, the digit will always be 2 and the remainder $5^n \cdot r_n$ will always equal 0.5. Thus, the base 5 representation of 0.5 is $\boxed{0.\overline{2}}$, where the digit 2 repeats forever.

7. Given the figure at right with m $\overset{\frown}{AB}= x°$, m $\overset{\frown}{BC}= 2x + 9°$, m $\overset{\frown}{CD}= x - 6°$, and m $\overset{\frown}{AD}= 3x + 21°$, find the measure of $\angle 1$ in degrees.



Solution: The sum of the measures of the four arcs $\overset{\frown}{AB}, \overset{\frown}{BC}, \overset{\frown}{CD}$ and $\overset{\frown}{AD}$ is equal to 360° because they form a circle. Write the equation and solve for x:

$$\begin{aligned} \text{m } \overset{\frown}{AB} + \text{m } \overset{\frown}{BC} + \text{m } \overset{\frown}{CD} + \text{m } \overset{\frown}{AD} &= x + (2x + 9) + (x - 6) + (3x + 21) \\ &= x + 2x + x + 3x + 9 - 6 + 21 = 7x + 24 = 360 \end{aligned}$$

Then $7x = 360 - 24 = 336$ and $x = \frac{336}{7} = 48$.

Recall that the measure of an angle formed by the intersection of two chords inside a circle is equal to the average (half the sum) of the measures of the two intercepted arcs. In this case, the chords are $\overline{AC}$ and $\overline{BD}$ and the arcs are $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$. Thus, $m\angle 1 = \frac{\text{m}\overset{\frown}{AB} + \text{m}\overset{\frown}{CD}}{2} = \frac{x + x - 6}{2} = \frac{2(48) - 6}{2} = 48 - 3 = \boxed{45}°$.

8. The following expression is equal to $\frac{a\sqrt{b}}{c}$ when written in simplest radical form.

$$5\sqrt{\frac{2}{9}} + 3\sqrt{\frac{9}{2}} - 5\sqrt{8}$$

Find the ordered triple (a, b, c) where $c > 0$.

Solution:

$$\begin{aligned}
5\sqrt{\frac{2}{9}} + 3\sqrt{\frac{9}{2}} - 5\sqrt{8} &= 5\frac{\sqrt{2}}{\sqrt{9}} + 3\frac{\sqrt{9}}{\sqrt{2}} - 5\sqrt{4\cdot 2} \\
&= \frac{5\sqrt{2}}{3} + \frac{3\cdot 3}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} - 5\sqrt{4}\cdot\sqrt{2} \\
&= \frac{5}{3}\cdot\sqrt{2} + \frac{9}{2}\cdot\sqrt{2} - 5\cdot 2\sqrt{2} \\
&= \left(\frac{5}{3} + \frac{9}{2} - 10\right)\sqrt{2} \\
&= \frac{5\cdot 2 + 9\cdot 3 - 10\cdot 6}{6}\cdot\sqrt{2} \\
&= \frac{10 + 27 - 60}{6}\cdot\sqrt{2} = \frac{-23\sqrt{2}}{6}
\end{aligned}$$

and $(a, b, c) = \boxed{(-23, 2, 6)}$.

9. Express the following expression as a single trigonometric function of x:

$$2\sin\frac{x}{2}\cos\frac{x}{2}\cot x$$

Solution: As a first step, recall the double angle formula for $\sin x$: $\sin 2\theta = 2\sin\theta\cos\theta$. Substitute $\theta = \frac{x}{2}$ into this formula yields $\sin 2\frac{x}{2} = \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$. Now, apply that identity, together with the identity $\cot x = \frac{\cos x}{\sin x}$, to simplify the original expression:

$$\begin{aligned}
2\sin\frac{x}{2}\cos\frac{x}{2}\cot x &= \sin x\cot x \\
&= \sin x\frac{\cos x}{\sin x} \\
&= \boxed{\cos x}
\end{aligned}$$